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Green's function for a Schrödinger operator and some related summation formulas

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Abstract

Summation formulas are obtained for products of associated Lagurre polynomials by means of the Green's function K for the Hamiltonian $H_0 = -\frac{d^2}{dx^2} + x^2 + Ax^{-2} \quad (A > 0)$. K is constructed by an application of a Mercer type theorem that arises in connection with integral equations. The new approach introduced in this paper may be useful for the construction of wider classes of generating function.

Keywords Schroödinger operators, singular potentials, Mercer's Theorem, Laguerre polynomials, Green's functions.

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1. Introduction and main results

Since the early development of quantum mechanics, highly singular potentials have attracted much attention. Two main reasons for this are (1) regular perturbation theory can fail badly for such potentials, and (2) in physics one often encounters phenomenological potentials that are strongly singular at the origin such as certain types of nucleon-nucleon interaction, and singular models of fields in arbitrary dimensions. A specific family of singular quantum Hamiltonians known as generalized harmonic oscillators given by

$$H(\lambda) = H_0 + \frac{\lambda}{x^\alpha} = -\frac{d^2}{dx^2} + x^2 + \frac{A}{x^2} + \frac{\lambda}{x^\alpha}, \quad (A \geq 0,\ \alpha > 0,\ \lambda \geq 0) \tag{1}$$

and acting in the Hilbert space $L_2(0,\infty)$ have been subject to intensive investigation recently. For a background and brief history of these problems we refer the reader to the summary in reference [1]. We have shown [1-2] that the set of eigenfunctions of $H_0 = H(0)$, namely

$$\psi_n(x) \equiv (-1)^n \sqrt{\frac{2\ (\gamma)_n}{n!\Gamma(\gamma)}} x^{\gamma-1/2} e^{-\frac{x^2}{2}} {}_1F_1(-n;\gamma;x^2) \text{ with } \gamma \equiv 1 + \frac{1}{2}\sqrt{1+4A} \quad (n = 0,1,2,\ldots), \tag{2}$$

constitutes an orthonormal basis for the Hilbert space $L_2(0,\infty)$. Here ${}_1F_1$ stands for the confluent hypergeometric function defined in terms of the associated Laguerre polynomials $L_n^{\gamma-1}(z)$ by

$${}_1F_1(-n;\gamma;z) = \frac{n!}{(\gamma)_n} L_n^{\gamma-1}(z). \tag{3}$$

This basis has proven to be useful in providing a complete variational study [1-5] of the spectrum of $H(\lambda)$ for arbitrary fixed $A \geq 0$, and $\lambda,\ \alpha > 0$. The advantage over earlier studies in the Hermite basis $A = 0$ was that for $A > 0$ the H_0-basis itself derives from a singular problem with the term A/x^2. In the present article, we explore another aspect of this basis. We shall prove that the eigenfunctions $\psi_n(x)$ satisfy the following identity:

$$\sum_{n=0}^{\infty} \frac{\psi_n(x)\psi_n(y)}{4n+2\gamma} = \begin{cases} w(x)v(y) & \text{for } 0 \leq y \leq x \\ v(x)w(y) & \text{for } 0 \leq x \leq y. \end{cases} \tag{4}$$

where

$$w(x)v(y) = 2^{-1}\ \sqrt{xy}\ K_\nu\left(\frac{x^2}{2}\right) I_\nu\left(\frac{y^2}{2}\right), \qquad \text{where} \nu = \frac{1}{2}(\gamma - 1).$$

In particular, we have, for $x = y$, that

$$\sum_{n=0}^{\infty} \frac{|\ \psi_n(x)\ |^2}{4n+2\gamma} = 2^{-1}\ x\ K_\nu\left(\frac{x^2}{2}\right) I_\nu\left(\frac{x^2}{2}\right). \tag{5}$$

Here $I_\nu(x)$ and $K_\nu(x)$ are modified Bessel functions of the first and second kind respectively. There are direct applications for these idenities. An obvious application

is that it can been seen as complimantary identity for Watson's famous result [6] (see also [7, p. 140, formula 14])

$$\sum_{n=0}^{\infty}\frac{(\gamma)_n}{n!(1+n)}{}_1F_1(-n;\gamma;x){}_1F_1(-n;\gamma;y)={}_1F_1(1;\gamma;y)\left[\Gamma(\gamma-1)\ x^{1-\gamma}e^x-\frac{1}{\gamma-1}{}_1F_1(1;\gamma;x)\right], \quad (6)$$

valid for $x \geq y > 0$. Second, it can be use in theory of coherent states to provide, for example, normalization factors of new class of coherent states labeled by confluent hypergeometric functions. Third, there are standard techniques known for generating closed form sums for products of hypergeometric functions and related polynomials, such as Laguerre polynomials. Srivastava *et al* [8-9] have discussed many different techniques that can be used for such purposes. It is noteworthy that the use of a kernel of a differential equation and a Mercer type theorem is not an idea that has been well explored in this context. It is our goal in the present article to show the usefulness of this approach to the construction of generating functions.

In order to prove our main results, we organize the paper as follows. In Section 2, we introduce two linearly independent solutions of the second-order homogeneous differential equation $H_0u=[-\frac{d^2}{dx^2}+(x^2+Ax^{-2})]u=0$. In Section 3, we construct the Green's function of H_0 and study some of its properties. The majorization of the Kernel operator $K(x,y)$ is investigated in Section 4. In Section 5, we introduce and prove a Mercer type theorem that allows us to conclude the absolute and uniformal convergence of the kernel $K(x,y)$ on the Hilbert space $L_2(0,\infty)$, and consequently prove our main results Theorem 1 and Theorem 2 from which formulas (4) and (5) follows immediately.

2. Second-order differential equation and its solutions

If we set $u(z)=z^{\alpha}\psi(z)$ with $z=\frac{1}{2}x^2$, we can easily show $H_0u=0$ reduces to

$$\frac{d^2\psi}{dz^2}+\frac{(2\alpha+\frac{1}{2})}{z}\frac{d\psi}{dz}-\left[1+\frac{4\alpha(\alpha-\frac{1}{2})-A}{4z^2}\right]\psi(z)=0. \quad (1)$$

If we adjust α so that $\psi(z)$ satifies the modified Bessel function

$$\frac{d^2\psi}{dz^2}+z^{-1}\frac{d\psi}{dz}-\left[\nu^2z^{-2}+1\right]\psi(z)=0,$$

we obtain, for $\alpha=\frac{1}{4}$ and $\nu=\frac{1}{4}\sqrt{1+4A}$, and from the basis $\{\sqrt{x}I_\nu(\frac{x^2}{2}),\sqrt{x}K_\nu(\frac{x^2}{2})\}$, the two linearly independent solutions

$$v(x)=B\sqrt{x}I_\nu(\frac{x^2}{2}),\quad w(x)=C\sqrt{x}K_\nu(\frac{x^2}{2}) \quad (2),$$

where B and C are constants to be determined. We note that, the Wronskian of $I_\nu(z)$ and $K_\nu(z)$ satisfies [6, p. 80, formula (90)]

$$I_\nu(z){K_\nu}'(z)-{I_\nu}'(z)K_\nu(z)=-z^{-1},$$

which equation we divide by $I_\nu{}^2(z)$ and thereby obtain the derivative of the quotient $(K_\nu/I_\nu)(z)$, namely

$$\left(\frac{K_\nu}{I_\nu}\right)'(z) = z^{-1}I_\nu{}^{-2}(z).$$

We integrate this expression from z to ∞ (using the properties $K_\nu(\infty) = 0$ and $I_\nu(\infty) = \infty$) and thus we arrive at

$$\left(\frac{K_\nu}{I_\nu}\right)(z) = \int_z^\infty \frac{1}{\xi I_\nu{}^2(\xi)}d\xi. \tag{3}$$

On the other hand, we make the replacements $z = x^2/2$, $K_\nu(x^2/2) = C^{-1}x^{-1/2}w(x)$ and $I_\nu(x^2/2) = B^{-1}x^{-1/2}v(x)$ in the immediately-preceeding formula, and find

$$\frac{K_\nu(x^2/2)}{I_\nu(x^2/2)} = \frac{B}{C} \times \frac{w(x)}{v(x)} = \int_{x^2/2}^\infty \frac{1}{\xi I_\nu{}^2(\xi)}d\xi,$$

where in the last integral we make the substitution $I_\nu(\xi) = B^{-1}(2\xi)^{-1/4}v(\sqrt{2\xi})$, thereby giving us

$$\frac{B}{C} \times \frac{w(x)}{v(x)} = \int_{\xi=x^2/2}^\infty \xi^{-1}\left[\frac{v(\sqrt{2\xi})}{B(\sqrt{2\xi})}\right]^{-2} d\xi = 2B^2\int_{\xi=x^2/2}^\infty \frac{1}{2\xi v^2(\sqrt{2\xi})(2\xi)^{-1/2}}d\xi = 2B^2\int_{r=x}^\infty \frac{1}{v^2(r)}dr.$$

The last integral expression was obtained via the substitution $r = \sqrt{2\xi}$. Equating the first expression with the last in the above equations leads to

$$\frac{w(x)}{v(x)} = 2BC\int_{r=x}^\infty \frac{1}{v^2(r)}dr \text{ or equivalently } w(x) = 2BCv(x)\int_{r=x}^\infty \frac{1}{v^2(r)}dr. \tag{4}$$

However, reduction of the order of the original differential equation implies that

$$v(x)\int_{r=x}^\infty \frac{1}{v^2(r)}dr$$

is the other solution of $H_0u = 0$, independent of $v(x)$, which result lets us conclude that $2BC = 1$ or $BC = 1/2$.

3. Mapping Properties of the Green's Function in $L_2(0,\infty)$

For the operator

$$H_0 = -\frac{d^2}{dx^2} + (x^2 + Ax^{-2}), \tag{1}$$

the linear space $D(H_0)$, consisting of all functions $u \in C^2(0,\infty) \cap C[0,\infty)$ with $u(0) = 0$, becomes a normed linear space by setting $\parallel u \parallel_\infty \equiv sup \mid u([0,\infty)) \mid$. For the Green's function of the linear transformation $H_0 : D(H_0) \mapsto C[0,\infty)$, we now formulate a well known result from the theory of ordinary differential equations, which may be easily arrived at from the properties of Green's function as stated in [11-13].

Lemma 1. *The differential operator* H_0 *maps* $D(H_0)$ *bijectively onto* $C[0,\infty)$ *by means of* ${H_0}^{-1} : f \mapsto u$ *, where*

$$({H_0}^{-1})f(x) = u(x) \equiv w(x) \int_{\xi=0}^{x} v(\xi) f(\xi) d\xi + v(x) \int_{\xi=x}^{\infty} w(\xi) f(\xi) d\xi, \tag{2}$$

with $\| {H_0}^{-1} f \|_\infty \leq 4A^{-1/2} \| f \|_\infty$.

Proof: The boundedness of ${H_0}^{-1}$ is simply a consequence of the inequality

$$v(x) \int_{\xi=x}^{\infty} [v(\xi)]^{-2} d\xi \leq \frac{\sqrt{2}}{a} [v(x)]^{-1} \tag{3}$$

where $a \equiv \sqrt[4]{A}$, the value of x where the potential $q_A(x) = x^2 + Ax^{-2}$ assumes its minimum. Indeed, since the function $q_A(x)$ assumes its minimum value $q_A(\sqrt[4]{A}) = 2\sqrt{A} = 2a^2$ at the point $x = a \equiv \sqrt[4]{A}$, we note that

$$v(x) = v(x') + (x - x')v'(x') + \int_{\xi=x'}^{x} (x - \xi)\ q_A(\xi)\ v(\xi)\ d\xi \quad \text{for} \quad 0 \leq x' \leq x,$$

and from this directly deduce

$$v(x) \geq v(x') + 2a^2 \int_{\xi=x'}^{x} (x - \xi)\ v(\xi)\ d\xi \quad \text{for} \quad 0 \leq x' \leq x.$$

By substituting $v(\xi) \geq v(x') + 2a^2 \int_{\eta=x'}^{\xi} (\xi - \eta) v(\eta) d\eta$ into the previous integral inequality, we further arrive at

$$v(x) \geq \left\{ 1 + \frac{1}{2!} 2a^2 (x - x')^2 \right\} v(x') + \frac{1}{3!} (2a^2)^2 \int_{\xi=x'}^{x} (x - \xi)^3\ v(\xi)\ d\xi \quad \text{for} \quad 0 \leq x' \leq x.$$

Iterating this procedure leads to $v(x) \geq v(x') \cosh \sqrt{2} a(x - x')$ for $x \geq x' \geq 0$. In particular

$$v(x) \geq 2^{-1} \exp(\sqrt{2} a(x - x')) v(x') \text{ for } x \geq x' \text{ and } v(x) \geq 2^{-1} \exp(\sqrt{2} a(x - a)) v(a) \text{ for } x \geq a,$$

which can also be written in the form $v(x') \leq 2 \exp(-\sqrt{2} a(x - x')) v(x)$ for $x \geq x' \geq 0$ and this in turn leads to

$$\int_{\xi=a}^{x} v(\xi) d\xi \leq \frac{\sqrt{2}}{a} v(x).$$

This result is also evident from the integration of the inequality $v(\xi) \leq 2 \exp(-\sqrt{2} a(x - \xi)) v(x)$ with respect to ξ on the interval (a, x). As a result of $v(\xi) \geq 2^{-1} \exp(\sqrt{2} a(\xi - x)) v(x)$ for all $\xi \geq x$, or by means of $[v(\xi)]^{-2} \leq 4 \exp(2\sqrt{2} a(x - \xi)) [v(x)]^{-2}$, leads to the inequality

$$\int_{\xi=x}^{\infty} [v(\xi)]^{-2} d\xi \leq 4 \int_{\xi=x}^{\infty} \exp(2\sqrt{2} a(x - \xi)) d\xi \times [v(x)]^{-2} \leq \frac{\sqrt{2}}{a} [v(x)]^{-2},$$

which proves (3.3). The injective nature of ${H_0}^{-1} : C[0,\infty) \mapsto D(H_0)$ is demonstrated as follows. Let $u(x)$ be a solution of $H_0 u = 0$ with $u(0) = 0$, whence $u(x) = Bw(x) + Cv(x)$.

On account of the asymptotic behavior of $w(x)$ and $v(x)$ as $x \to 0^+$, namely $w(x)$ and $v(x) \to \infty$ and 0 respectively as $x \to 0^+$, combined with $\| u \|_\infty < \infty$ leads to $B = 0$, that is to say $u(x) = Cv(x)$; however, the asymptotic behavior of $v(x)$, namely

$$v(x) = \frac{1}{\sqrt{\pi x}} e^{\frac{x^2}{2}} [1 + O(x^2)]$$

as $x \to \infty$, implies that $C = 0$. Thus $H_0 : D(H_0) \mapsto C[0, \infty)$ is bijective. This completes the proof. □

We have therefore that the Green's function $K(x, y)$ of the differential operator $H_0 : D(H_0) \mapsto C[0, \infty)$, which permits us to write the action of ${H_0}^{-1}$ in terms of integration on $[0, \infty)$, has the integral representation

$$ {H_0}^{-1} f(x) = \int_0^\infty K(x, y)\ f(y)\ dy, \tag{4}$$

where

$$K(x, y) = \begin{cases} w(x)v(y), & \text{for } 0 \le y \le x, \\ \\ v(x)w(y), & \text{for } 0 \le x \le y. \end{cases} \tag{5}$$

It thus becomes evident that the kernel $K(x, y)$ is a continuous non-negative function on $[0, \infty) \times [0, \infty)$ because, for $\nu = \frac{1}{2}(\gamma - 1) = \frac{1}{4}\sqrt{1 + 4A}$, $\sqrt{x} I_\nu(x^2/2)$ and $\sqrt{x} K_\nu(x^2/2)$ are continuous on $[0, \infty)$ and $(0, \infty)$ respectively, and, furthermore, both are positive on $(0, \infty)$. Thus $K(x, y)$ is continuous on $[0, \infty) \times [0, \infty) \setminus \{(0, 0)\}$.

For the continuity at $(0, 0)$ we again turn to the asymptotic behavior of $w(x)$ and $v(x)$ as $x \to 0^+$. In the lower sector $\{(x, y) : 0 \le \text{arctg}(y/x) \le \pi/4\}$ of the first quadrant of the (x, y)-plane, we have that

$$\begin{aligned} K(x, y) &= w(x)v(y) \\ &= \frac{1}{4\pi\nu} x^{1/2 - 2\nu} \left[1 + O(x^\epsilon)\right] y^{1/2 + 2\nu} \left[1 + O(y^4)\right] \\ &= \frac{1}{4\pi\nu} (xy)^{1/2} (y/x)^{2\nu} \left[1 + O(y^4)\right] \left[1 + O(x^\epsilon)\right] \\ &\to 0 \text{ as } (x, y) \to (0, 0), \end{aligned}$$

because in this sector $\mid y/x \mid \le 1$. Consequently, $K(x, y)$ is continuous in the lower sector $\{(x, y) : 0 \le \text{arctg}(y/x) \le \pi/4\}$, whereas the symmetry of $K(x, y)$ - i. e. $K(x, y) = K(y, x)$ - guarantees the continuity in upper sector $\{(x, y) : \pi/4 \le \text{arctg}(y/x) \le \pi/2\}$ of the first quadrant of the (x, y)-plane.

We know further that $H_0 \equiv -d^2u/dx^2 + [x^2 + Ax^{-2}]u = 0 \quad (A > 0)$ is a symmetric lower semi-bounded operator in the Hilbert space $L_2(0, \infty)$ with domain of definition consisting of all $L_2(0, \infty)$-functions u vanishing at 0, having absolutely continuous derivative $u' \in L_2(0, \infty)$ on $[0, \infty)$ such that $[x^2 + Ax^{-2}]u(x)$ is also an $L_2(0, \infty)$-function in x. Since the linear manifold $C_0^\infty(0, \infty)$, of all complex valued infinitely differentiable

function on $(0, \infty)$ with compact support, lies dense in this domain of definition of H_0 as well as in $L_2(0, \infty)$, we readily conclude that the domain of definition of H_0 is also a dense subset of $L_2(0, \infty)$. Thus H_0 possesses a Friedrichs' extension, which extension we again denote by H_0, and this extension [14, Sec. 7.2-7.3]-[15, p. 335] is a self-adjoint operator in $L_2(0, \infty)$. For $H_0\psi_n = E_n\psi_n$ and because the orthonormalized set of eigenfunctions (1.2) with corresponding eigenvalues $E_n = 4n + 2\gamma = 4n + 2 + \sqrt{1 + 4A}$ of H_0 forms [1] a complete orthonormal set of functions of the Hilbert space $L_2(0, \infty)$, we shall have that the spectrum of this Friedrichs' extension H_0 is a purely point-spectrum, consisting only of the simple eigenvalues $E_n = 4n + 2\gamma = 4n + 2 + \sqrt{1 + 4A}$. Thus the spectral family $\{P_\mu : \mu \in \Re\}$ of H_0, which is an "increasing" projection-operator valued Saltus function on the set $\Re$ of real numbers [17, p. 92]-[15, Ch. I, Sec. 7], is

$$P_\mu \equiv \begin{cases} 0, & \text{for } -\infty < \mu < 2\gamma \\ \psi_0 \otimes \psi_0, & \text{for } 4(0) + 2\gamma \leq \mu < 4(1) + 2\gamma \\ \psi_0 \otimes \psi_0 + \psi_1 \otimes \psi_1, & \text{for } 4(1) + 2\gamma \leq \mu < 4(2) + 2\gamma \\ \psi_0 \otimes \psi_0 + \psi_1 \otimes \psi_1 + \psi_2 \otimes \psi_2, & \text{for } 4(2) + 2\gamma \leq \mu < 4(3) + 2\gamma \\ \dots & \\ \sum_{k=0}^{n-1} \psi_k \otimes \psi_k, & \text{for } 4(n-1) + 2\gamma \leq \mu < 4(n) + 2\gamma \\ \dots & \end{cases} \tag{6}$$

For any two $L_2(0, \infty)$-functions ψ and ϕ, the expression $\psi \otimes \phi$ denotes the operator of rank 1 defined by

$$(\phi \otimes \psi) f(x) \equiv <f \mid \psi> \phi(x) = \left[\int_{\xi=0}^{\infty} f(\xi) \overline{\psi(\xi)} d\xi \right] \times \phi(x) = \int_{\xi=o}^{\infty} (\phi \otimes \psi)(x, \xi) f(\xi) d\xi, \tag{7}$$

with L_2-kernel $(\phi \otimes \psi)(x, y) \equiv \phi(x)\overline{\psi(y)}$ on $L_2(0, \infty)$, which in our case turns out be an integral operator. It is further clear that the spectral family $\{P_\mu : \mu \in \Re\}$ is increasing in μ - i. e. $P_\lambda \leq P_\mu$ for $\lambda \leq \mu$ - as well as continuous from the right - i. e. $P_\mu = P_{\mu+0}$ - in the sense of strong convergence (denoted by $\rightarrow$) in the Hilbert space $L_2(0, \infty)$. Moreover, $P_\mu \rightarrow 0$ or I (the identity operator) according as $\mu \rightarrow -\infty$ or ∞. In consequence, the spectral decomposition [15, Sec. 120, p. 320] of our self-adjoint operator H_0 allows H_0 to be represented (as well as functions of H_0) as a Stieltjes integral of μ (functions of μ) with respect to the spectral family $\{P_\mu\}$ on the set $\Re = (-\infty, \infty)$ of real numbers. This turns out to be

$$H_0 = \int_{-\infty}^{\infty} \mu d_\mu P_\mu \tag{8}$$

with

$$D(H_0) = \left\{ f \in L_2(0, \infty) : \int_{-\infty}^{\infty} \mu^2 d_\mu \|P_\mu f\|^2 = \sum_{n=0}^{\infty} (4n + 2\gamma)^2 |<f|\psi_n>|^2 < \infty \right\}$$

and

$$H_0{}^{-1} = \int_{-\infty}^{\infty} \mu^{-1} d_\mu P_\mu = \sum_{n=0}^{\infty} (4n+2\gamma)^{-1}[P_{4n+2\gamma} - P_{4n+2\gamma-0}] = \sum_{n=0}^{\infty} (4n+2\gamma)^{-1}(\psi_n \otimes \psi_n), \quad (9)$$

where $P_{4n+2\gamma} - P_{4n+2\gamma-0} = \psi_n \otimes \psi_n$ is the projection onto the eigenspace $(L.H.)(\psi_n)$ spanned by the single eigenfunction ψ_n for each of the integers $n \geq 0$. The abbreviation $(L.H.)$ stands for "the linear hull of whatever is between the two brackets to its immediate right", and 'linear hull' means the set of all linear combinations. Consequently, the Green's function $K(x,y)$ of the differential operator H_0, namely the kernel of our self-adjoint operator H_0 restricted to the previous domain $D(H_0)$, takes on the form

$$K(x,y) = \sum_{n=0}^{\infty} (4n+2\gamma)^{-1}(\psi_n \otimes \psi_n)(x,y) = \sum_{n=0}^{\infty} \frac{\psi_n(x)\psi_n(y)}{4n+2\gamma} \text{ a. e. on } [0,\infty) \times [0,\infty) \quad (10)$$

with respect to Lebesgue measure on $[0,\infty)^2 = [0,\infty) \times [0,\infty)$, which shall turn out to be a positive L_2-kernel on $[0,\infty)$ with "finite double norm" [19, p. 13]. Herein we must emphasize the almost everywhere (a. e.) nature of the immediately-preceeding equality. The finite double norm of kernel $K(x,y)$ is defined as

$$||| K ||| \equiv \sqrt{\int_0^\infty \int_0^\infty | K(x,y) |^2 \, dydx} = \sqrt{\sum_{n=0}^{\infty} (4n+2\gamma)^{-2}} < \infty, \quad (10)$$

and consequently many of the ideas, but not all, that led to of Mercer's Theorem [19, p. 127] are applicable. However, because $\sum_{n=0}^{\infty} (4n+2\gamma)^{-1} = \infty$, it cannot be expected that all of the results of Mercer's Theorem carry over; specifically, K as an operator "on" the Hilbert space $L_2(0,\infty)$ fails to be an operator of trace class.

4. Majorization Property of the Kernel of Operator K on $L_2(0,\infty)$

Let us now call the extension of H_0 to all of $L_2(0,\infty)$ the operator K, which is an operator "on" $L_2(0,\infty)$ (instead of *in* $L_2(0,\infty)$). We may do this, on account [19, Th. 4.5.1, p. 63] of the fact that kernel $K(x,y)$ is an L_2-kernel and therefore the norm relation

$$\| K \| \leq ||| K ||| \equiv \sqrt{\int_0^\infty \int_0^\infty | K(x,y) |^2 \, dydx} = \sqrt{\sum_{n=0}^{\infty} (4n+2\gamma)^{-2}} < \infty, \quad (1)$$

guarantees that operator K has domain of definition $D(K) = L_2(0,\infty)$. Herein, $\| K \|$ and $||| K |||$ denote the operator norm of K and double norm of its kernel $K(x,y)$ respectively. The action of K on $L_2(0,\infty)$ is

$$(Kf)(x) = \int_0^\infty K(x,y)f(y)dy = \sum_{n=0}^{\infty} \frac{<f|\psi_n>}{4n+2\gamma}\psi_n(x) \text{ a. e. in } x \text{ on } [0,\infty), \quad (2)$$

and moreover, this operator K on $L_2(0,\infty)$ is also the Friedrichs' extension of H_0. We first consider the kernel $K(x,y)$ as the Green's function of H_0, and note that the completeness of the orthonormal basis $\{\psi_n : n \geq 0\}$, where the normalized eigenfunction ψ_n corresponds to the simple eigenvalue $\lambda_n = 4n + 2\gamma$, entails that

$$<Kf \mid g> = \sum_{n=0}^{\infty} \frac{<f \mid \psi_n><\psi_n \mid g>}{4n+2\gamma} \quad \text{for all } f \text{ and } g \in L_2(0,\infty). \tag{3}$$

In particular, we replace f and g herein by the sequence of $L_2(0,\infty)$-functions $\delta_n(x)$, tending weakly towards the Dirac-δ function $\delta(x)$, defined by

$$\delta_n(x) \equiv n\big[1 - n \mid x \mid \big] \quad \text{for } \mid x \mid \leq 1/n \text{ and } 0 \text{ otherwise on the set } \Re,$$

where $\Re$ is the set of real numbers. It becomes immediately clear, that out of $<Kf \mid f>$ always exceeding each of the finite sums $\sum_{k=0}^{N}[4n+2\gamma]^{-1} \mid <f \mid \psi_k> \mid^2$, the inequality

$$<K\delta_n(\cdot - x) \mid \delta_n(\cdot - x)> \geq \sum_{k=0}^{N} \frac{<\delta_n(\cdot - x) \mid \psi_n><\psi_n \mid \delta_n(\cdot - x)>}{4n+2\gamma} \text{ for all } N \geq 0$$

implies, by way of letting $n \to \infty$ and holding $x \geq 0$ fixed, that

$$K(x,x) \geq \sum_{k=0}^{N}[4n+2\gamma]^{-1} \mid \psi_k(x) \mid^2 \quad \text{for all } N \geq 0.$$

Therefore, it follows that

$$\sum_{n=0}^{\infty} \frac{\mid \psi_n(x) \mid^2}{4n+2\gamma} \leq K(x,x) \text{ for all } x \in [0,\infty),$$

where the function $K(x,x)$ has the precise form given by ($\nu = \frac{1}{4}\sqrt{1+4A}$)

$$K(x,x) = w(x)v(x) = 2^{-1}\ x\ I_\nu\left(\frac{x^2}{2}\right)K_\nu\left(\frac{x^2}{2}\right) \tag{4}$$

as well as asymptotic behavior as $x \to 0^+$ and ∞ respectively given by

$$K(x,x) = \begin{cases} \frac{1}{\sqrt{1+4A}}\ x\ \big[1 + O(x^\epsilon)\big], & \text{for } x \to 0^+ \\ 2^{-1}\ x^{-1}\ \big[1 + O(x^{-2})\big], & \text{for } x \to \infty. \end{cases} \tag{5}$$

5. Mercer Theorem Type Properties of the Kernel $K(x,y)$

From the asymptotic behavior (4.5) of the majorizing function $K(x,x)$ of $\sum_{n=0}^{\infty}[4n + 2\gamma]^{-1} \mid \psi_n(x) \mid^2$ as well as the continuity of $K(x,x)$ on $[0,\infty)$, we readily see that if we let $M_\infty \equiv \sup\{K(x,x) : x \in [0,\infty)\}$, the above asymptotic behavior guarantees that $M_\infty < \infty$, then

$$\sum_{n=0}^{\infty} \frac{\mid \psi_n(x) \mid^2}{4n+2\gamma} \leq M_\infty \text{ for all } x \in [0,\infty). \tag{1}$$

It therefore follows that $\sum_{n=0}^{\infty}[4n+2\gamma]^{-1} \mid \psi_n(x) \mid^2$ converges for all $x \in [0,\infty)$. Moreover, the Cauchy-Schwarz inequality allows us to write for all non-negative integers N that

$$\sum_{n=N}^{\infty} \frac{\mid \psi_n(x)\psi_n(y) \mid}{4n+2\gamma} \le \{ \sum_{n=N}^{\infty} \frac{\mid \psi_n(x) \mid^2}{4n+2\gamma} \}^{1/2} \times \{ \sum_{n=N}^{\infty} \frac{\mid \psi_n(y) \mid^2}{4n+2\gamma} \}^{1/2} \text{ for all } x,y \in [0,\infty). \quad (2)$$

It is further follows that the series $\sum_{n=0}^{\infty}[4n+2\gamma]^{-1}\psi_n(x)\psi_n(y)$ converges absolutely, because out of the immediately-preceeding inequality shall follow

$$\begin{aligned} \sum_{n=N}^{\infty} \frac{\mid \psi_n(x)\psi_n(y) \mid}{4n+2\gamma} &\le \{ \sum_{n=N}^{\infty} \frac{\mid \psi_n(x) \mid^2}{4n+2\gamma} \}^{1/2} \times \sqrt{M_\infty} \text{ for all } x \in [0,\infty) \text{ and} \\ \sum_{n=N}^{\infty} \frac{\mid \psi_n(x)\psi_n(y) \mid}{4n+2\gamma} &\le \{ \sum_{n=N}^{\infty} \frac{\mid \psi_n(y) \mid^2}{4n+2\gamma} \}^{1/2} \times \sqrt{M_\infty} \text{ for all } y \in [0,\infty). \end{aligned} \quad (3)$$

Let $\epsilon > 0$. It then follows that, for every $y \in [0,\infty)$, there exists an integer $N = N(y)$ such that

$$\sum_{n=N}^{\infty} \frac{\mid \psi_n(x)\psi_n(y) \mid}{4n+2\gamma} \le \epsilon\sqrt{M_\infty} \quad (4)$$

for all $x \in [0,\infty)$, provided $N = N(y)$ is chosen sufficiently large; and there exists correspondingly for every $x \in [0,\infty)$ an integer $N = N(y)$ such that

$$\sum_{n=N}^{\infty} \frac{\mid \psi_n(x)\psi_n(y) \mid}{4n+2\gamma} \le \epsilon\sqrt{M_\infty} \quad (5)$$

for all $x \in [0,\infty)$, provided $N = N(x)$ is chosen sufficiently large. These two statements (5.4) and (5.5) are valid because the series $\sum_{n=0}^{\infty}[4n+2\gamma]^{-1} \mid \psi_n(x) \mid^2$ is majorized by the constant M_∞ on $[0,\infty)$. Hence we are led to the following Mercer type theorem.

Theorem 1. *The kernel $K(x,y)$ of the operator K on the Hilbert space $L_2(0,\infty)$ with spectral decomposition $K = \sum_{n=0}^{\infty}[4n+2\gamma]^{-1}(\psi_n \otimes \psi_n)$ possesses the following property: the convergence of the series*

$$\sum_{n=0}^{\infty} \frac{\psi_n(x)\psi_n(y)}{4n+2\gamma} = K(x,y) \quad (6)$$

is absolute and uniform on every compact subset of $[0,\infty) \times [0,\infty)$.

Proof: The series $\sum_{n=0}^{\infty}[4n+2\gamma]^{-1} \mid \psi_n(x)\psi_n(y) \mid$ is uniformly convergent in x on $[0,\infty)$ for every $y \ge 0$, as well as uniformly convergent in y on $[0,\infty)$ for every $x \ge 0$, as evident from (5.4) and (5.5). Therefore $\sum_{n=0}^{\infty}[4n+2\gamma]^{-1}\psi_n(x)\psi_n(y)$ represents a continuous function in variable x on $[0,\infty)$ for every $y \ge 0$, as well as a continuous function in variable y on $[0,\infty)$ for every $x \ge 0$. For every $L_2(0,\infty)$-function f the Fourier expansion of Kf has the form

$$(Kf)(x) = \int_0^\infty K(x,y)f(y)dy = \sum_{n=0}^{\infty} \frac{<f|\psi_n>}{4n+2\gamma}\psi_n(x) \quad \text{a. e. in } x \text{ on } [0,\infty)$$

and possesses the following property

$$\begin{aligned}\sum_{n=N}^{\infty} \mid \frac{<f \mid \psi_n>}{4n+2\gamma}\psi_n(x) \mid &\leq \Big\{ \sum_{n=N}^{\infty} \mid <f|\psi_n> \mid^2 \Big\}^{1/2} \Big\{ \sum_{n=N}^{\infty} \frac{\mid \psi_n(x) \mid^2}{(4n+2\gamma)^2} \Big\}^{1/2} \\ &\leq \Big\{ \sum_{n=N}^{\infty} \mid <f|\psi_n> \mid^2 \Big\}^{1/2} \{ \sum_{n=N}^{\infty} \frac{\mid \psi_n(x) \mid^2}{(4n+2\gamma)} \}^{1/2} \\ &\leq \Big\{ \sum_{n=N}^{\infty} \mid <f|\psi_n> \mid^2 \Big\}^{1/2} \times \sqrt{M_\infty},\end{aligned}$$

and therefore the sum $\sum_{n=0}^{\infty}[4n+2\gamma]^{-1}<f|\psi_n>\psi_n(x)$ is an absolutely and uniformly convergent series of continuous functions on $[0,\infty)$ whose limit is continuous on $[0,\infty)$, and hence

$$(Kf)(x) = \int_0^\infty K(x,y)f(y)dy = \sum_{n=0}^{\infty} \frac{<f|\psi_n>}{4n+2\gamma}\psi_n(x) \text{ for all } x \in [0,\infty).$$

Since in the series $\sum_{n=0}^{\infty}[4n+2\gamma]^{-1}(\psi_n \otimes \psi_n)$ the set of kernels $\{(\psi_n \otimes \psi_n) : \ n \geq 0\}$ constitutes an orthonormal set of $L_2(0,\infty)$-kernels, the Riesz-Fischer Theorem, combined with $K(x,y) = \sum_{n=0}^{\infty}[4n+2\gamma]^{-1}(\psi_n \otimes \psi_n)(x,y)$ a. e. on $[0,\infty) \times [0,\infty)$ and the fact that $\sum_{n=0}^{\infty}[4n+2\gamma]^{-2} < \infty$, leads to

$$||| \, K - \sum_{n=0}^{\infty} \frac{\psi_n \otimes \psi_n}{4n+2\gamma} \, |||= \{ \int_0^\infty \int_0^\infty |K(x,y) - \sum_{n=0}^{\infty} \frac{\psi_n(x)\psi_n(y)}{4n+2\gamma}|^2 dydx \}^{1/2} = 0$$

and furthermore to

$$\int_0^\infty \left[K(x,y) - \sum_{n=0}^{\infty} \frac{\psi_n(x)\psi_n(y)}{4n+2\gamma} \right] f(y)dy = 0 \text{ for all } f \in L_2(0,\infty).$$

Now let C denote any compact interval $[a,b]$ contained in $[0,\infty)$, and consider the immediately-preceeding equality for all functions continuous on C and vanishing on $[0,\infty) \setminus C$. Note that these functions belong to $L_2(0,\infty)$. This in turn implies that

$$\int_C \left[K(x,y) - \sum_{n=0}^{\infty} \frac{\psi_n(x)\psi_n(y)}{4n+2\gamma} \right] f(y)dy = 0$$

for all $x \in C$ and f continuous on C with $f([0,\infty) \setminus C) = 0$. By choosing f for any arbitrary, however momentarily, fixed $x \in C$ as follows,

$$f(y) = K(x,y) - \sum_{n=0}^{\infty} \frac{\psi_n(x)\psi_n(y)}{4n+2\gamma} \text{ and } f([0,\infty) \setminus C) \equiv 0,$$

where for this fixed $x \in C$ the series represents a continuous function in variable y on set C vanishing on $[0,\infty) \setminus C$, and substituting it into the immediately above equality, we obtain that

$$\int_C \mid K(x,y) - \sum_{n=0}^{\infty} \frac{\psi_n(x)\psi_n(y)}{4n+2\gamma} \mid^2 \, dy = 0 \text{ for all } x \in C.$$

Because the integrand above is a continuous function of y on the compact subset C, which is an arbitrary finite closed interval contained in $[0,\infty)$, we obtain

$$K(x,y) - \sum_{n=0}^{\infty} \frac{\psi_n(x)\psi_n(y)}{4n+2\gamma} = 0 \text{ for all } x,y \in [0,\infty),$$

which, for $x = y \in [0,\infty)$, specifically yields

$$K(x,x) = \sum_{n=0}^{\infty} \frac{\mid \psi_n(x) \mid^2}{4n+2\gamma} \text{ for all } x \in [0,\infty).$$

We now invoke Dini's Theorem, which states: Every monotone sequence of real valued continuous functions on a compact metric space with continuous limit, converges uniformly to its limit. Hence by Dini's Theorem [17, p. 66] the convergence of $K(x,x) = \sum_{n=0}^{\infty}[4n+2\gamma]^{-1} \mid \psi_n(x) \mid^2$ is therefore uniform on every compact subset of $[0,\infty)$, and therefore the series $\sum_{n=0}^{\infty}[4n+2\gamma]^{-1}\psi_n(x)\psi_n(y)$ converges absolutely and uniformly on every compact subset of $[0,\infty)\times[0,\infty)$ with limit $K(x,y)$ - i. e. $K(x,y) = \sum_{n=0}^{\infty}[4n+2\gamma]^{-1}\psi_n(x)\psi_n(y)$ for all $(x,y) \in [0,\infty)\times[0,\infty)$. This completes the proof. $\square$

Again we note that this is only a Mercer type theorem, because it makes no conclusion about the operator K or its $L_2(0,\infty)$-kernel $K(x,y)$ being of trace class; whereas Mercer's theorem makes an affirmative statement [15, pp. 245-246], see also [16]-[19], concerning the trace class nature of operator K. As consequence of this Mercer type theorem for our Green's function $K(x,y)$, we return to the property of the spectral decomposition of the operator K discussed before, because any complex valued function $W(\mu)$ of the real variable μ on $(-\infty,\infty)$ determines [15, Sec. 127 and 128] an operator $W(K)$ *in* $L_2(0,\infty)$ defined by

$$W(K) = \int_{-\infty}^{\infty} W(\mu) d_\mu P_\mu = \sum_{n=0}^{\infty} W(4n+2\gamma)[P_{4n+2\gamma} - P_{4n+2\gamma-0}] = \sum_{n=0}^{\infty} W(4n+2\gamma)(\psi_n \otimes \psi_n),$$

whose domain of definition $D(W(K))$ consists of all $L_2(0,\infty)$-functions satisfying

$$\int_{-\infty}^{\infty} \mid W(\mu) \mid^2 d_\mu \langle P_\mu f | f \rangle = \int_{-\infty}^{\infty} \mid W(\mu) \mid^2 d_\mu \parallel P_\mu f \parallel^2 = \sum_{n=0}^{\infty} \mid W(4n+2\gamma) \mid^2 \mid \langle f|\psi_n \rangle \mid^2 < \infty.$$

It is clearly evident that the operators $W(K)$ are always densely defined, regardless of the function W considered on $\Re = (-\infty,\infty)$, because every domain of definition $D(W(K))$ always contains $(L.H.)(\psi_n(n \geq 0))$. Thus we have that the inverse $[\lambda I - K]^{-1}$ of the operator $\lambda I - K$, in the normed algebra of bounded linear operators on $L_2(0,\infty)$, comes about from the complex valued function $W(\mu) = [\lambda - \mu]^{-1}$ of the real variable μ and takes the form

$$\begin{aligned} [\lambda I - K]^{-1} &= \int_{-\infty}^{\infty} [\lambda - \mu]^{-1} d_\mu P_\mu \\ &= \sum_{n=0}^{\infty} [\lambda - (4n+2\gamma)]^{-1}[P_{4n+2\gamma} - P_{4n+2\gamma-0}] \\ &= \sum_{n=0}^{\infty} [\lambda - 4n - 2\gamma]^{-1}(\psi_n \otimes \psi_n), \end{aligned}$$

provided $\lambda \notin \sigma(K) = \{4n+2\gamma : n \text{ a non-negative integer}\}$, namely the spectrum of operator K in the algebra of bounded linear operators on $L_2(0,\infty)$, with continuous kernel

$$[\lambda - \cdot]^{-1}(x,y) = \sum_{n=0}^{\infty} \frac{\psi_n(x)\psi_n(y)}{\lambda - 4n - 2\gamma} \text{ for all } (x,y) \in [0,\infty)\times[0,\infty),$$

where the convergence is absolute on $[0,\infty)\times[0,\infty)$ and uniform on every compact subset of $[0,\infty)\times[0,\infty)$. We may consequently summarize our results in the following two theorems.

Theorem 2. *The* $C[0,\infty)$ *-function* $K(x,x)$ *arising out of the Green's function of the differential operator* $H_0 = -\frac{d^2}{dx^2} + x^2 + Ax^{-2} \quad (A>0)$ *satisfies:*

$$\sum_{n=0}^{\infty} \frac{|\ \psi_n(x)\ |^2}{4n+2\gamma} = K(x,x) = w(x)v(x) = 2^{-1}\ x\ I_{\frac{1}{2}(\gamma-1)}\left(\frac{x^2}{2}\right) K_{\frac{1}{2}(\gamma-1)}\left(\frac{x^2}{2}\right)$$

where $\gamma = 1 + \frac{1}{2}\sqrt{1+4A}$, *with uniform convergence on every compact subset of* $[0,\infty)$. $K(x,x)$ *has the asymptotic behaviour*

$$K(x,x) = \begin{cases} \frac{1}{\sqrt{1+4A}}\ x\ \left[1 + O(x^{\epsilon})\right], & \text{for } x \to 0^+ \\ 2^{-1}\ x^{-1}\ \left[1 + O(x^{-2})\right], & \text{for } x \to \infty. \end{cases}$$

Theorem 3. *The continuous kernel* $K(x,y)$ *on* $[0,\infty)\times[0,\infty)$, *arising from the Green's function of the differential operator* $H_0 = -\frac{d^2}{dx^2} + x^2 + Ax^{-2} \quad (A>0)$, *satisfies:*

$$K(x,y) = \sum_{n=0}^{\infty} \frac{\psi_n(x)\psi_n(y)}{4n+2\gamma} = \begin{cases} w(x)v(y) & \text{for } 0 \le y \le x \\ v(x)w(y) & \text{for } 0 \le x \le y. \end{cases}$$

on $[0,\infty)\times[0,\infty)$ *with convergence being absolute and uniform on every compact subset of* $[0,\infty)\times[0,\infty)$, *where*

$$w(x)v(y) = 2^{-1}\ \sqrt{xy}\ K_{\frac{1}{2}(\gamma-1)}\left(\frac{x^2}{2}\right)\ I_{\frac{1}{2}(\gamma-1)}\left(\frac{y^2}{2}\right).$$

We further conclude that, for the orthonormal basis $\{\psi_n(x) : n \ge 0\}$, (1.2), of the Hilbert space $L_2(0,\infty)$, the two new summation formulas (1.4) and (1.5) follows immediately.

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